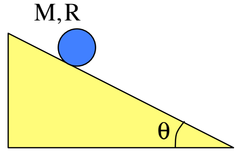
Problem 10.57

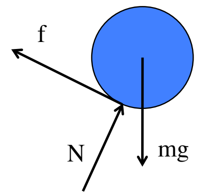
A disk rolls down an incline.



a.) Derive an expression for the *acceleration* of its *center of mass*.

This is a N.S.L. problem. In fact, there are two ways to do this, one in which we look at the translational and rotational motion of and about the body's *center of mass*, and one in which we look at the system as though the mass was executing a pure rotation about its contact point. I'll first do the most obvious, which is the former. I will show the least obvious as extra at the end.

We don't want the disk to slip, so there has to be friction, but in cases like this where we have a rolling object there is both rolling friction and static friction. As we will never be given the coefficient of friction in these problems, the bottom line is that we have to deal with friction as an unknown with which to be dealt.

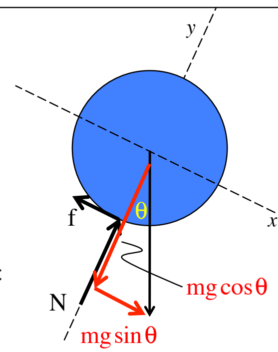


Doing a knee-jerk summing of the forces, we get:

$\underline{\sum F_y:}$

$$N - mg\cos\theta = ma_y$$

$$\Rightarrow \quad N = mg\cos\theta$$



Unfortunately, as the frictional force is a little bit static and a little bit rolling (and as we have no μ_s), this is useless (hence the "knee-jerk"). Continuing:

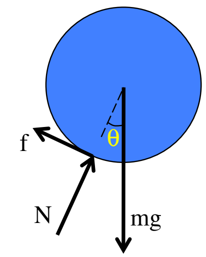
$\underline{\sum F_x:}$

$$-f + mg\sin\theta = ma$$

$$\Rightarrow \quad f = mg\sin\theta - ma$$

We don't know the *acceleration* or the *frictional force* so we need another expression. That will come from summing the torques about the disk's *center of mass*.

When summing the torques, we don't need components so a more useful f.b.d. is shown to the right. The torque about the *center of mass* due to "N" and "mg" are both zero as they both pass through the point about which we are taking the torque. With that, the rotational N.S.L. expression becomes:



$\underline{\sum \Gamma_{cm}:}$

$$-fR = -I_{cm}\alpha$$

Note: Why the negative sign in front of "fR?" A couple of reasons. If you do the right-hand rule appropriate for *cross products*, you get a vector *into* the page. OR, if you notice the frictional force makes the body want to angular accelerate *clockwise*. Any of those will give you a negative torque. As for the negative sign in front of the acceleration term, again, the body is angularly accelerating *clockwise*. That is a negative rotational vector (also, the only torque term on the left side of the equation is negative, so the *angular acceleration* term on the right better be negative, also. Otherwise, you have an equation that isn't an equality.

Note that the *moment of inertia* for a disk about its *center of mass* is $I_{cm} = \frac{1}{2}mR^2$.

Remember that the relationship between the *acceleration* of a rolling body's *center of mass* and *angular acceleration around that center of mass* is $a_{cm} = R\alpha$. (The justification of this follows the same line as was presented in the last part of the last problem in showing that for a rolling object, $v_{cm} = R\omega$.)

With all that, we can combine the two derived relationships and write:

$$-fR = -I_{cm}\alpha$$

$$\Rightarrow \quad f\cancel{R} = \left(\frac{1}{2}m\cancel{R^2}\right)\left(\frac{a}{\cancel{R}}\right)$$

$$\Rightarrow \quad f = \frac{1}{2}ma$$

With $f = mg\sin\theta - ma$, we can combine (eliminating the "f's," and write:

$$\frac{1}{2}ma = mg\sin\theta - ma$$

$$a = \frac{2}{3}g\sin\theta$$

b.) How would the system have differed if the body had been a *hoop* of the same mass and radius?

Before looking at the math, what would we expect conceptually?

With all the mass at the rim of the hoop, its *moment of inertia* would have been bigger than the disks (more mass is distributed, on average, farther from the central axis). With the same torque being applied in both cases, a larger *moment of inertia* would means a smaller *angular acceleration*. That's what we would expect. This is the same conclusion the text's Solution Manual came to, but they did it by actually doing the math (it's the same problem with a different *moment of inertia* . . . which is not something you have to do unless you don't see the conceptual side of it).

c.) What is the minimum *coefficient of frictional* required to maintain rolling without slipping?

Assuming all the friction is static, we can write:

$$\begin{aligned} f_s &= \mu_s N \\ &= \mu_s mg\cos\theta \end{aligned}$$

From our torque calculation, we have additionally derived the relationship:

$$\begin{aligned} f_s &= \frac{1}{2}ma \\ &= \frac{1}{2}m\left(\frac{2}{3}g\sin\theta\right) \\ &= \frac{1}{3}mg\sin\theta \end{aligned}$$

Equating the two frictional force expressions, we get:

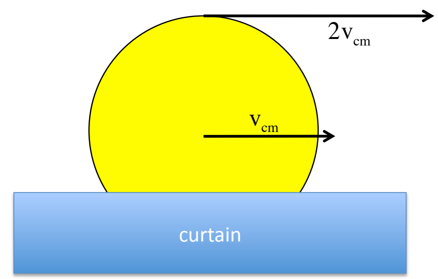
$$\mu_s mg\cos\theta = \frac{1}{3}mg\sin\theta$$

$$\Rightarrow \quad \mu_s = \left(\frac{1}{3}\right)g\frac{\sin\theta}{\cos\theta} = \frac{1}{3}g\tan\theta$$

FINI! The following is extra.

EXTRA: At the end of the last problem, I pointed out the motion of a body that is rolling looks different from different coordinate axes, but that the *angular velocity* of the mass of material about any axis on the structure will be the same. There are some interesting consequences to that that I didn't point out.

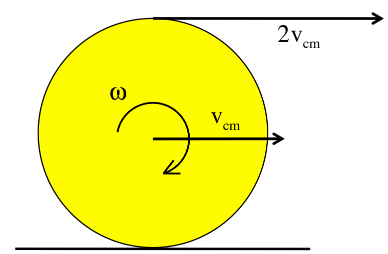
Let's say you get a very quick glimpse of an object that is partially hidden behind a curtain. In the instant you get your view, you see the top part moving faster than the center of mass, as shown below.



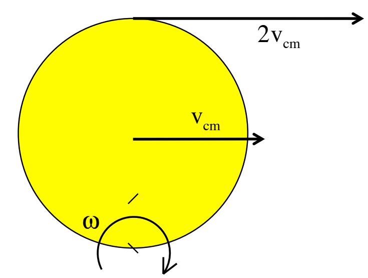
SO WHAT'S HAPPENING BEHIND THE CURTAIN?

One of two things could be happening. One possibility is that of a disk rolling on a surface. The other is that of a disk pinned at one edge that is executing a pure rotation about that point. (See sketches.)

To possibilities:



rolling

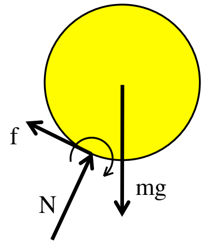


pinned
(pure rotation)

WHAT'S INTERESTING IS THAT *INSTANTANEOUSLY*, THERE IS NO DIFFERENCE BETWEEN THE PHYSICAL MANIFESTATIONS OF THE TWO SITUATIONS.

So let's rethink our incline problem. If we think of the motion as an instantaneous, pure rotation about the contact point (see sketch), we can sum the torques about that point and . . . well, you'll see!

Calling the point of contact *point P*, the torques due to "f" and "N" will both be zero as they pass through that point, and the only torque acting is due to gravity. Using the $r_\perp$ approach, we get simply:

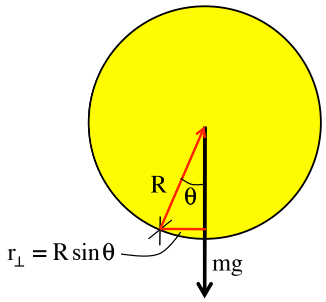


$$\underline{\sum \Gamma_p :}$$

$$-\left|\vec{F}_g\right| r_\perp = -I_p \alpha$$

$$-(mg)(R\sin\theta) = -I_p \alpha$$

Because we have eliminated the second major unknown (that of friction—it's torque is zero), all we have to do is use the Parallel Axis Theorem to determine I_p, use the ever present $a = R\alpha$ and we are done!



With the *moment of inertia* about the *center of mass* being

$$I_{cm} = \frac{1}{2} mR^2$$

The *Parallel Axis Theorem* yields a *moment of inertia* about *point P* as:

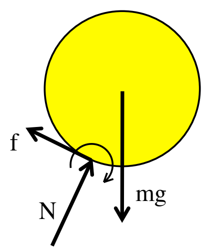
$$I_p = I_{cm} + md^2$$

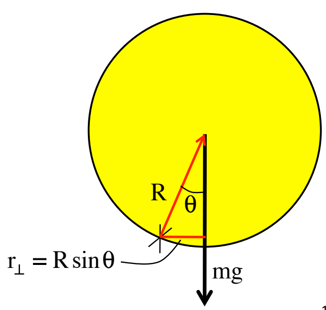
$$= \frac{1}{2} mR^2 + mR^2$$

$$= \frac{3}{2} mR^2$$

Remembering that the *angular acceleration* is related to the acceleration of the *center of mass* by:

$$\alpha = \frac{a_{cm}}{R}$$





We can write (or re-write) that torque expression in a more complete form as:

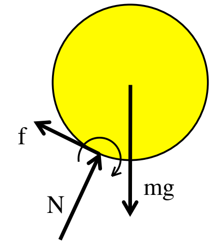
$$\underline{\sum \Gamma_p :}$$

$$-mgR\sin\theta = -I_p \alpha$$

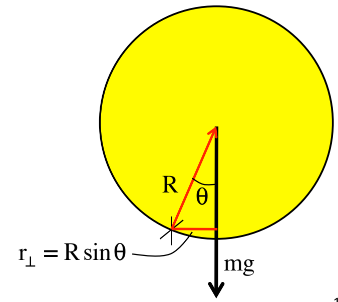
$$\Rightarrow \quad mgR\sin\theta = \left(\frac{3}{2} mR^2\right)\left(\frac{a_{cm}}{R}\right)$$

$$\Rightarrow \quad g\sin\theta = \frac{3}{2} a_{cm}$$

$$\Rightarrow \quad a_{cm} = \frac{3}{2} g\sin\theta$$



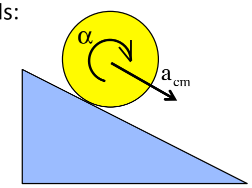
This is the same relationship for "a" that we derived back on *page 4*, except all we had to do here was sum the torques about the instantaneous fixed point, use the *Parallel Axis Theorem*, and we're done.



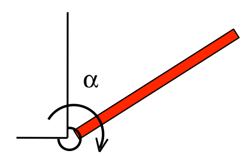
So if there are two ways to do these problems, which way should one choose.

My advise is to look to see what the system is actually doing, then use the approach that most natural fits the system. In other words:

1.) If it is a system like a ball rolling down an incline, a situation in which there is clearly acceleration *of* the *center of mass* along with angular acceleration *around* the *center of mass*, then use the *center of mass* approach (i.e., the one we originally used).



2.) If, on the other hand, the system is like a pinned beam rotating about one end where there is clearly a "rotation about a fixed point," then use the fixed point approach.



Both approaches will ALWAYS work, but in most cases, one will be easier to negotiate than the other (trying, for instance, to deal with a pinned beam by looking at the *acceleration* of its *center of mass*, etc., is a HUGE hassle!). In the end, though, it is your choice.